1) a) $\tilde{X} = X + aN$.

Recall $N_u = a_{11}X_u + a_{21}X_v$

$N_v = a_{12}X_u + a_{22}X_v$

where $a_{11} = \dfrac{fF - eG}{EG - F^2}$, $a_{21} = \dfrac{eF - fE}{EG-F^2}$, and $K = a_{11}a_{22} - a_{12}a_{21}$

$a_{12} = \dfrac{gF - fG}{EG-F^2}$, $a_{22} = \dfrac{fF - gE}{EG - F^2}$. $H = -\frac{1}{2}(a_{11} + a_{22})$.

since $S_p = -dN$.

$\tilde{X}_u = X_u + aN_u = X_u + a(a_{11}X_u + a_{21}X_v)$

$= (1 + aa_{11})X_u + aa_{21}X_v$.

$\tilde{X}_v = X_v + aN_v = X_v + a(a_{12}X_u + a_{22}X_v)$

$= aa_{12}X_u + (1 + aa_{22})X_v$.

So $\tilde{X}_u \times \tilde{X}_v = (1 + aa_{11})(1 + aa_{22})X_u \times X_v - aa_{21}aa_{12}X_u \times X_v$

$= [1 + 2a(a_{11} + a_{22}) + (a^2 a_{11}a_{22} - a^2 a_{21}a_{12})]X_u \times X_v$

$= (1 - 2Ha + Ka^2)X_u \times X_v$, as required.

b) Note that $|\tilde{X}_u \times \tilde{X}_v| = (1 - 2Ha + Ka^2)|X_u \times X_v|$, so

$$\tilde{N} = \frac{\tilde{X}_u \times \tilde{X}_v}{|\tilde{X}_u \times \tilde{X}_v|} = \frac{\cancel{(1-2Ha+Ka^2)}X_u \times X_v}{\cancel{(1-2Ha+Ka^2)}|X_u \times X_v|} = N.$$

So $\tilde{N}$ and N are parallel, so $d\tilde{N}_p(\tilde{X}_u) = dN_p(X_u)$, $d\tilde{N}_p(\tilde{X}_v) = dN_p(X_v)$.

so $N_u = \tilde{N}_u = -S_p(X_u)$, $N_v = \tilde{N}_v = -S_p(X_v)$

$$[S_p] = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = \begin{bmatrix} 1+aa_{11} & aa_{21} \\ aa_{12} & 1+aa_{22} \end{bmatrix} \begin{bmatrix} -\tilde{a}_{11} & -\tilde{a}_{12} \\ -\tilde{a}_{21} & -\tilde{a}_{22} \end{bmatrix}$$

by $\tilde{X}_u = (1+aa_{11}) X_u + aa_{21} X_v$

$\tilde{X}_v = aa_{12} X_u + (1+a_{22}) X_v$

$$= [I - aS_p][\tilde{S}_p].$$

$$\Rightarrow [\tilde{S}_p] = [I - aS_p]^{-1} [S_p] \quad (*)$$

Diagonalize $[S_p] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then by $(*)$, the eigenvalues of $[\tilde{S}_p]$ are $\dfrac{\lambda_1}{1-a\lambda_1}$, $\dfrac{\lambda_2}{1-a\lambda_2}$.

So
$$\tilde{K} = \frac{\lambda_1}{1-a\lambda_1} \cdot \frac{\lambda_2}{1-a\lambda_2} = \frac{\lambda_1\lambda_2}{(1-a\lambda_1)(1-a\lambda_2)} = \frac{\lambda_1\lambda_2}{1-a\lambda_1-a\lambda_2+a^2\lambda_1\lambda_2}$$
$$= \frac{K}{1-2Ha+Ka^2}.$$

$$\tilde{H} = \frac{1}{2}\left(\frac{\lambda_1}{1-a\lambda_1} + \frac{\lambda_2}{1-a\lambda_2}\right) = \frac{1}{2}\left(\frac{\lambda_1(1-a\lambda_2)+\lambda_2(1-a\lambda_1)}{(1-a\lambda_1)(1-a\lambda_2)}\right)$$
$$= \frac{1}{2}\left(\frac{\lambda_1 - a\lambda_1\lambda_2 + \lambda_2 - a\lambda_1\lambda_2}{1-a\lambda_1-a\lambda_2+a^2\lambda_1\lambda_2}\right) = \frac{H-Ka}{1-2Ha+Ka^2}$$

2) Note: established in earlier tutorial that a closed bounded surface $S \subseteq \mathbb{R}^3$ has at least one elliptic point (ie. $\exists p \in S$ s.t. $K(p) > 0$).
Suppose in addition that S is minimal. Then in particular at p, $H(p) = 0$. So letting k_1, k_2 be the principal curvatures at p,

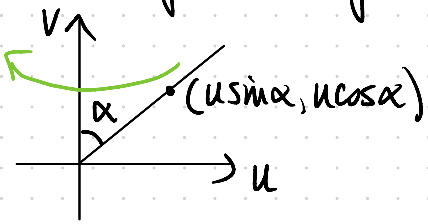
$$\underset{(1)}{k_1 k_2 > 0} \quad \text{and} \quad \underset{(2)}{\tfrac{1}{2}(k_1 + k_2) = 0}.$$

but this leads to a contradiction since (1) implies k_1, k_2 have the same sign while (2) implies k_1, k_2 have opposite signs. $\lightning$

3) $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ by $\quad U = \{(u,v) : u>0\}$.

$F(u,v) = (u\sin\alpha\cos v,\ u\sin\alpha\sin v,\ u\cos\alpha)$ $\quad \alpha = \text{const}.$

a) Observe that F indeed parameterizes the cone C with vertex at the origin and angle 2α (compare with Q2 of HW3).



To show that it is a local diffeomorphism, it suffices to show dF has full rank

$$F_u = (\sin\alpha\cos v,\ \sin\alpha\sin v,\ \cos\alpha)$$
$$F_v = (-u\sin\alpha\sin v,\ u\sin\alpha\cos v,\ 0)$$

which are lin. indep. for $u \neq 0$.

Hence by IFT, F is a local diffeomorphism onto its image.

b) The cone C param. by F has the 1st F.F. given by

$$E = 1,\ F = 0,\ G = u^2\sin^2\alpha.$$

We can param. U in local coordinates by

$X(\rho, \theta) = (\rho\cos\theta,\ \rho\sin\theta,\ 0)$ where $\rho = u\sin\alpha > 0$, $0 < \theta < 2\pi\sin\alpha$.

Then 1st F.F. given by X is given by

$$\tilde{E} = 1,\ \tilde{F} = 0,\ \tilde{G} = \rho^2 = u^2\sin^2\alpha.$$

So by Prop 1. in Sec 4-2 of do Carmo, the map $F \circ X^{-1}$ is a local isometry.

4) By isothermal param. $g_{ij} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $g^{-1}_{ij} = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$

We use

$$K = \frac{1}{2}\left(g^{ij}\left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^p_{ij}\Gamma^k_{kp} - \Gamma^p_{ik}\Gamma^k_{jp}\right)\right)$$

$$= \frac{1}{2} g^{11}\left(\partial_k \Gamma^k_{11} - \partial_1 \Gamma^k_{1k} + \Gamma^p_{11}\Gamma^k_{kp} - \Gamma^p_{1k}\Gamma^k_{1p}\right)) \quad (g^{12}, g^{21} = 0)$$

$$+ \frac{1}{2} g^{22}\left(\partial_k \Gamma^k_{22} - \partial_2 \Gamma^k_{2k} + \Gamma^p_{22}\Gamma^k_{kp} - \Gamma^p_{2k}\Gamma^k_{2p}\right))$$

Where $\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\partial_1 g_{11} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}}\right) = \frac{1}{2}\frac{\lambda_u}{\lambda}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11}\left(\cancelto{0}{\partial_1 g_{21}} + \partial_2 g_{11} - \cancelto{0}{\partial_1 g_{12}}\right) = \frac{1}{2}\frac{\lambda_v}{\lambda}$$

$$\Gamma^1_{21} = \frac{1}{2} g^{11}\left(\partial_2 g_{11} + \cancelto{0}{\partial_1 g_{21}} - \cancelto{0}{\partial_1 g_{21}}\right) = \frac{1}{2}\frac{\lambda_v}{\lambda}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\cancelto{0}{\partial_2 g_{21}} + \cancelto{0}{\partial_2 g_{21}} - \partial_1 g_{22}\right) = -\frac{1}{2}\frac{\lambda_u}{\lambda}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22}\left(\cancelto{0}{\partial_1 g_{12}} + \cancelto{0}{\partial_1 g_{12}} - \partial_2 g_{11}\right) = -\frac{1}{2}\frac{\lambda_v}{\lambda}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}\left(\partial_1 g_{22} + \cancelto{0}{\partial_2 g_{12}} - \cancelto{0}{\partial_2 g_{12}}\right) = \frac{1}{2}\frac{\lambda_u}{\lambda}$$

$$\Gamma^2_{21} = \frac{1}{2} g^{22}\left(\cancelto{0}{\partial_2 g_{12}} + \partial_1 g_{22} - \cancelto{0}{\partial_2 g_{12}}\right) = \frac{1}{2}\frac{\lambda_u}{\lambda}$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22}\left(\partial_2 g_{22} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}}\right) = \frac{1}{2}\frac{\lambda_v}{\lambda}$$

$$K = \frac{1}{2}\left(g^{ij}\left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^p_{ij}\Gamma^k_{kp} - \Gamma^p_{ik}\Gamma^k_{jp}\right)\right)$$

$$= \frac{1}{2}\left(g^{11}\left(\partial_k \Gamma^k_{11} - \partial_1 \Gamma^k_{1k} + \Gamma^p_{11}\Gamma^k_{kp} - \Gamma^p_{1k}\Gamma^k_{1p}\right)\right)$$

$$+ \frac{1}{2}\left(g^{22}\left(\partial_k \Gamma^k_{22} - \partial_2 \Gamma^k_{2k} + \Gamma^p_{22}\Gamma^k_{kp} - \Gamma^p_{2k}\Gamma^k_{2p}\right)\right)$$

$$= \frac{1}{2} g^{11}\left(\partial_k \Gamma^k_{11} - \partial_1 \Gamma^k_{1k} + \Gamma^1_{11}\Gamma^k_{k1} + \Gamma^2_{11}\Gamma^k_{k2} - \Gamma^1_{1k}\Gamma^k_{11} - \Gamma^2_{1k}\Gamma^k_{12}\right)$$

$$+ \frac{1}{2} g^{22}\left(\partial_k \Gamma^k_{22} - \partial_2 \Gamma^k_{2k} + \Gamma^1_{22}\Gamma^k_{k1} + \Gamma^2_{22}\Gamma^k_{k2} - \Gamma^1_{2k}\Gamma^k_{21} - \Gamma^2_{2k}\Gamma^k_{22}\right)$$

$$= \frac{1}{2} g^{11}\Big(\cancel{\partial_1 \Gamma^1_{11}} + \partial_2 \Gamma^2_{11} - \cancel{\partial_1 \Gamma^1_{11}} - \partial_1 \Gamma^2_{12} + \cancel{\Gamma^1_{11}\Gamma^1_{11}} + \Gamma^1_{11}\Gamma^2_{21}$$
$$+ \cancel{\Gamma^2_{11}\Gamma^1_{12}} + \Gamma^2_{11}\Gamma^2_{22} - \cancel{\Gamma^1_{11}\Gamma^1_{11}} - \Gamma^1_{12}\Gamma^2_{11}$$
$$- \cancel{\Gamma^2_{11}\Gamma^1_{12}} - \Gamma^2_{12}\Gamma^2_{12}\Big)$$

$$+ \frac{1}{2} g^{22}\Big(\partial_1 \Gamma^1_{22} + \cancel{\partial_2 \Gamma^2_{22}} - \partial_2 \Gamma^1_{21} - \cancel{\partial_2 \Gamma^2_{22}} + \Gamma^1_{22}\Gamma^1_{11} + \cancel{\Gamma^1_{22}\Gamma^2_{21}}$$
$$+ \Gamma^2_{22}\Gamma^1_{12} + \cancel{\Gamma^2_{22}\Gamma^2_{22}} - \Gamma^1_{21}\Gamma^1_{21} - \Gamma^1_{22}\Gamma^2_{21} - \cancel{\Gamma^2_{21}\Gamma^1_{22}}$$
$$- \cancel{\Gamma^2_{22}\Gamma^2_{22}}\Big)$$

$$= \frac{1}{2} g^{11}\left(\partial_2 \Gamma^2_{11} - \partial_1 \Gamma^2_{12} + \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12}\right)$$

$$+ \frac{1}{2} g^{22}\left(\partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{21}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^2_{21}\right)$$

$$
\begin{aligned}
&= \tfrac{1}{2}\lambda^{-1}\Bigg(-\tfrac{1}{2}\left(\frac{\lambda\lambda_{vv}-(\lambda_v)^2}{\lambda^2}\right)-\tfrac{1}{2}\left(\frac{\lambda\lambda_{uu}-(\lambda_u)^2}{\lambda^2}\right)+\left(\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)^2+\left(-\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)\left(\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right) \\
&\qquad -\left(\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)\left(-\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)-\left(\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)\left(\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)\Bigg) \\
&\quad + \tfrac{1}{2}\lambda^{-1}\Bigg(-\tfrac{1}{2}\left(\frac{\lambda\lambda_{uu}-(\lambda_u)^2}{\lambda^2}\right)-\tfrac{1}{2}\left(\frac{\lambda\lambda_{vv}-(\lambda_v)^2}{\lambda^2}\right)+\left(-\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)\left(\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)+\left(\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)^2 \\
&\qquad -\left(\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)\left(\tfrac{1}{2}\frac{\lambda_v}{\lambda}\right)-\left(-\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)\left(\tfrac{1}{2}\frac{\lambda_u}{\lambda}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}\lambda^{-1}\Bigg(-\tfrac{1}{2}\frac{\lambda\lambda_{vv}}{\lambda^2}+\tfrac{1}{2}\frac{\lambda_v^2}{\lambda^2}-\tfrac{1}{2}\frac{\lambda\lambda_{uu}}{\lambda^2}+\tfrac{1}{2}\frac{\lambda_u^2}{\lambda^2}+\cancel{\tfrac{1}{4}\frac{\lambda_u^2}{\lambda^2}}-\cancel{\tfrac{1}{4}\frac{\lambda_v^2}{\lambda^2}} \\
&\qquad +\cancel{\tfrac{1}{4}\frac{\lambda_v^2}{\lambda^2}}-\cancel{\tfrac{1}{4}\frac{\lambda_u^2}{\lambda^2}}-\tfrac{1}{2}\frac{\lambda\lambda_{uu}}{\lambda^2}+\tfrac{1}{2}\frac{\lambda_u^2}{\lambda^2}-\tfrac{1}{2}\frac{\lambda\lambda_{vv}}{\lambda^2}+\tfrac{1}{2}\frac{\lambda_v^2}{\lambda^2} \\
&\qquad -\cancel{\tfrac{1}{4}\frac{\lambda_u^2}{\lambda^2}}+\cancel{\tfrac{1}{4}\frac{\lambda_v^2}{\lambda^2}}-\cancel{\tfrac{1}{4}\frac{\lambda_v^2}{\lambda^2}}+\cancel{\tfrac{1}{4}\frac{\lambda_u^2}{\lambda^2}}\Bigg)
\end{aligned}
$$

$$
= \tfrac{1}{2}\lambda^{-1}\left(-\frac{\lambda\lambda_{vv}}{\lambda^2}-\frac{\lambda\lambda_{uu}}{\lambda^2}+\frac{\lambda_u^2}{\lambda^2}+\frac{\lambda_v^2}{\lambda^2}\right)
$$

$$
= \frac{-1}{2\lambda}\Delta\log\lambda
$$

as required